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Dilute Birman–Wenzl–Murakami algebra and $D^{(2)}_{n+1}$ models

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Abstract. A 'dilute' generalization of the Birman–Wenzl–Murakami algebra is considered. It can be 'Baxterized' to a solution of the Yang–Baxter algebra. The $D^{(2)}_{n+1}$ vertex models constitute a series of solvable lattice models which realize this algebraic structure. They can be regarded as a dilute version of the $B^{(1)}_n$ vertex models.

1. Introduction

The theory of two-dimensional solvable lattice models is intimately connected with a list of algebraic structures with a wide range of applications in mathematics and physics [1]. Among those are, for instance, the braid group [2] and the Temperley–Lieb [3] and Hecke algebras [4]. The braid and Temperley–Lieb or monoid [5] operators were combined into a single (so-called braid–monoid) algebra by Birman and Wenzl [6] and independently by Murakami [7] (see also [8]). Besides being closely related to solvable lattice models, these algebras have another important property: they admit a simple diagrammatic interpretation in terms of transformations of strands or strings and are of importance in the theory of knot and link invariants (see e.g. [8]). Recently, a generalization of braid–monoid algebras has been introduced [9] which amounts to considering strings of different 'colours'. These are connected with recently constructed critical solvable lattice models [10–13] which are related to (coloured) dense or dilute loop models.

In what follows, let us briefly recollect the basic definitions. A braid–monoid algebra (also called knit or tangle algebra) is defined as the algebra generated by b_j, b_j^{-1} and e_j ($1 \leqslant j \leqslant N-1$, where N corresponds to the number of strings in the diagrammatic interpretation mentioned above) subject to the following list of relations:

$$\begin{aligned} & b_j b_j^{-1} = b_j^{-1} b_j = I \\ & b_j b_k = b_k b_j \qquad \text{for } |j-k|>1 \\ & b_j b_{j+1} b_j = b_{j+1} b_j b_{j+1} \end{aligned} \tag{1.1}$$

$$\begin{aligned} & e_j^2 = \sqrt{Q}\, e_j \\ & e_j e_k = e_k e_j \qquad \text{for } |j-k|>1 \\ & e_j e_{j\pm1} e_j = e_j \end{aligned} \tag{1.2}$$

$$\begin{aligned} & b_j e_j = e_j b_j = \omega e_j \\ & b_j e_k = e_k b_j \qquad \text{for } |j-k|>1 \\ & b_{j\pm1} b_j e_{j\pm1} = e_j b_{j\pm1} b_j = e_j e_{j\pm1}. \end{aligned} \tag{1.3}$$

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Here $\sqrt{Q}$ and ω are central elements (hence numbers in any irreducible representation) and I denotes the identity. Equations (1.1) are the braid relations, and equations (1.2) are the defining relations of the Temperley–Lieb algebra [3]. Usually, two equations are added to these relations, which are

$$f(b_j) = 0 \qquad g(b_j) = e_j \tag{1.4}$$

where f and g are some polynomials. The algebra originally investigated in [6, 7] (the Birman–Wenzl–Murakami algebra), for example, corresponds to the case where f is a cubic and g a quadratic polynomial in the braids. The reason why these equations are listed separately is that they do not have a diagrammatic representation and that one might consider these as properties of certain representations rather than as part of the defining relations of the algebra.

Solvable lattice models are commonly constructed as vertex or as face or IRF (interaction-round-a-face) [14] models whose Boltzmann weights satisfy the Yang–Baxter equation. This property can equivalently be stated in the form that they give a representation of the Yang–Baxter algebra [14–16] defined by

$$\begin{aligned} &X_j(u)X_{j+1}(u+v)X_j(v) = X_{j+1}(v)X_j(u+v)X_{j+1}(u) \\ &X_j(u)X_k(v) = X_k(v)X_j(u) \qquad \text{for } |j-k| > 1 \end{aligned} \tag{1.5}$$

where $X_j(u)$ (u denotes the spectral parameter) are local operators whose matrix elements are the Boltzmann weights of the model (see, e.g. [8] for details). For vertex models, these Yang–Baxter operators (also called 'local face operators') are particularly simple as they act on an N-fold tensor space (N being the number of vertices in one row), acting as the R matrix (to be precise, as $\check{R}(u) = PR(u)$ where P is the permutation map $P : v \otimes w \mapsto w \otimes v$) at slots j and $j+1$ and as the identity elsewhere.

Every crossing-symmetric (see e.g. [8]) representation of the Yang–Baxter algebra yields a representation of the braid–monoid algebra by setting

$$e_j = X_j(\lambda) \qquad b_j^{\pm 1} = \kappa^{\pm 1} \lim_{u \to \mp i\infty} \frac{X_j(u)}{\varrho(u)} \tag{1.6}$$

where λ is the crossing parameter and κ and $\varrho(u)$ are appropriately chosen normalization factors. Conversely, one may be able to 'Baxterize' [17] a representation of the braid–monoid algebra to a representation of the full Yang–Baxter algebra. This is especially useful if one can find a general expression for the Yang–Baxter operator in terms of the braids and monoids which can be shown to fulfill the Yang–Baxter algebra as a consequence of the algebraic relations alone (maybe apart from some additional assumptions, for instance, about the polynomial reduction relations (1.4)). In this way, each appropriate representation of the braid–monoid algebra gives rise to a solvable lattice model.

This paper is organised as follows. To start with, we give a short summary of the Birman–Wenzl–Murakami case which corresponds to a braid–monoid algebra where the braids satisfy a cubic reduction relation. Representations of this algebra occur in the $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, and $A_n^{(2)}$ series of vertex models [18, 19] and associated face models. This is of course well known [20, 21], but still there are a few surprising observations which arise. In section 3, we introduce the dilute generalization of the Birman–Wenzl–Murakami algebra and mention its graphical interpretation in terms of diagrams acting on strings of two kinds. This algebra is then 'Baxterized' [17] to a solution of the Yang–Baxter algebra (1.5) in section 4. Here, the corresponding examples of known models are the $D_{n+1}^{(2)}$ vertex models [19]. The associated representation of the dilute algebra can be regarded as a dilute version of the Birman–Wenzl–Murakami algebra related to the $B_n^{(1)}$ models, which becomes more transparent by a suitable change of basis in the expression of the R matrix in [19]. Finally, the results are summarized in section 5.

2. Birman–Wenzl–Murakami algebra

We assume that the braid satisfies the cubic

$$(b_j - \sigma^{-1}I)(b_j + \sigma I)(b_j - \sigma\tau^2 I) = 0 \tag{2.1}$$

where the third eigenvalue is the twist ω and hence

$$\begin{aligned} \omega &= \sigma\tau^2 \\ \sqrt{Q} &= 1 + \frac{\omega - \omega^{-1}}{\sigma - \sigma^{-1}} \\ e_j &= I + \frac{(b_j - b_j^{-1})}{\sigma - \sigma^{-1}}. \end{aligned} \tag{2.2}$$

The braid–monoid algebra (1.1)–(1.3) with these additional relations is known as the Birman–Wenzl–Murakami (BWM) algebra. Then it is easy to show that the following ansatz satisfies the Yang–Baxter algebra (1.5) [21]:

$$X_j(u) = I \ + \ \zeta^{-1}\,\eta^{-1}\,(z - z^{-1})\,(\tau^{-1}z\,b_j \ - \ \tau z^{-1}\,b_j^{-1}). \tag{2.3}$$

Here $z = \exp(\mathrm{i}u)$ (u denotes the spectral parameter), $\zeta = (\sigma - \sigma^{-1})$, and $\eta = (\tau - \tau^{-1})$. It is a crossing symmetric with crossing parameter λ given by $\tau = \exp(\mathrm{i}\lambda)$ (note that $X_j(\lambda) = e_j$), and satisfies the inversion relation

$$X_j(u)X_j(-u) = \varrho(u)\,\varrho(-u)\,I \tag{2.4}$$

with

$$\varrho(u) = \zeta^{-1}\,\eta^{-1}\,(\sigma z^{-1} - \sigma^{-1}z)\,(\tau z^{-1} - \tau^{-1}z). \tag{2.5}$$

Examples of solvable lattice models which can be expressed in this form [20, 21] are given by the $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, and $A_n^{(2)}$ vertex models [18, 19] and related face models [20]. In the notation of [19] (where the R matrices are parametrized by $x = z^2$ and a complex parameter k), the corresponding values of σ and τ are

$$(\sigma, \tau^2) = \begin{cases} (k, \xi) & \text{for } B_n^{(1)} \text{ and } D_n^{(1)} \\ (-k^{-1}, \xi) & \text{for } C_n^{(1)} \text{ and } A_n^{(2)} \end{cases} \tag{2.6}$$

where, as in [19], $\xi = k^{2n-1}, k^{2n+2}, k^{2n-2}, -k^{n+1}$ for $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ and $A_n^{(2)}$, respectively.

An interesting observation is in order. None of the above expressions in the BWM algebra are altered by interchanging $\sigma \leftrightarrow -\sigma^{-1}$. This means that for a given representation there are in fact *two* Yang–Baxter operators (which *a priori* are not the same, since the values of τ are different), the second one being given by equation (2.3) with

$$\left(\sigma, \tau^2\right) \longrightarrow \left(\sigma', \tau'^2\right) = \left(-\sigma^{-1}, -\sigma^2\tau^2\right). \tag{2.7}$$

Having a closer look at the examples provided by the vertex models, one realizes that the pairs $(A_{2n}^{(2)}, B_n^{(1)})$ and $(A_{2n-1}^{(2)}, D_n^{(1)})$ are built on identical representations of the BWM algebra. This implies that face models associated to $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$ can be directly deduced from the corresponding $B_n^{(1)}$ [22] and $D_n^{(1)}$ models, respectively. Note that the $A_{2n-1}^{(2)}$ face models of [22] are actually built on $C_n^{(1)}$ and not on $D_n^{(1)}$ (in contrast to the vertex models of [18, 19]), compare the discussion in [22] at the end of section 1.

Of course, one cannot obtain the recently constructed dilute A–D–E models [10, 11] (which are related to the $A_2^{(2)}$ (Izergin–Korepin [23]) R matrix) in this way since these have a different algebraic structure (corresponding to a different gauge of the $A_2^{(2)}$ R matrix, see [9] for details) as already mentioned above.

Surprisingly, there is no obvious partner for the $C_n^{(1)}$ models among the list of known solvable models. This either means that the second solution defines a new additional series of solvable vertex models (and corresponding face models) or that they are related to other models, for instance by a gauge transformation. This question certainly deserves further clarification.

3. Dilute braid–monoid algebra

The idea of considering multi-colour generalizations of braid–monoid algebras originates in the investigation of recently constructed face models [10–12] which are related to (coloured) loop models. In [9], it was shown that these models could be conveniently described in terms of two-colour generalizations of the Temperley–Lieb algebra [3]. This has been the motivation to look for similar generalizations of the BWM algebra and associated solvable lattice models.

To generate the m-colour algebra, we need 'coloured' braid and monoid operators $b_j^{\pm(a,b)}$, $e_j^{(a,b)}$ (where $a, b = 1, 2, \ldots, m$ denote the colours) as well as projectors $P_j^{(a)}$ which project onto colour a at position j. Also, $\sqrt{Q^{(a)}}$ and $\omega^{(a)}$ become the colour-dependent 'Temperley–Lieb eigenvalue' and twist. Note that here we use superscripts '+' and '−' to distinguish coloured braids and 'inverse' braids (see [9] for details). The full set of relations which defines the algebra (for the general m-colour case) can also be found in [9]; they are straightforward generalizations of the one-colour relations (1.1)–(1.3). In complete analogy to the one-colour case, all the relations can be interpreted graphically where one has to consider strings of two different colours (see [9]) which can never join.

Here, we are only interested in a 'dilute' (two-colour) braid–monoid algebra by which we mean a two-colour case where one colour (we choose colour '2') is trivial in the sense that

$$b_j^{\pm(2,2)} = e_j^{(2,2)} = P_j^{(2)} P_{j+1}^{(2)} \tag{3.1}$$

which implies $\sqrt{Q^{(2)}} = 1$ and $\omega^{(2)} = 1$. Moreover,

$$b_j^{+(a,b)} = b_j^{-(a,b)} = b_j^{(a,b)} \qquad (a \neq b) \tag{3.2}$$

wherefore we drop the superscript $\pm$ for the mixed braids. This means that the only non-trivial operators acting on two sites j and $j+1$ are $b_j^{\pm(1,1)}$, $e_j^{(1,1)}$, $p_j^{(a,b)} = P_j^{(a)} P_{j+1}^{(b)}$ $(a, b \in \{1, 2\})$, $b_j^{(a,\tilde{a})}$, and $e_j^{(a,\tilde{a})}$ $(a \in \{1, 2\}, \tilde{a} = 3 - a)$.

Thinking in terms of the graphical representation, this means that the second colour can also be interpreted as a vacancy of a string — but it may be easier to draw pictures with two types of strings as one has to keep in mind where these vacancies are. Still, the special properties of the second colour lead to a somewhat simplified graphical representation than for the full two-colour algebra (equation (3.2) for instance means that one does not have to distinguish between two types of crossings of strings of different kind), see [24].

4. Baxterization of dilute BWM algebra

We now consider a dilute Birman–Wenzl–Murakami algebra as introduced in the preceding section where the subalgebra generated by objects of colour '1' is of Birman–Wenzl–Murakami type. Changing our notation of section 2 slightly, we assume the following cubic relation for the braids $b_j^{+(1,1)}$:

$$\left(b_j^{+(1,1)} - \sigma^{-1} p_j^{(1,1)}\right)\left(b_j^{+(1,1)} + \sigma p_j^{(1,1)}\right)\left(b_j^{+(1,1)} - \tau^2 p_j^{(1,1)}\right) = 0. \tag{4.1}$$

Here the third eigenvalue is again the twist $\omega^{(1)}$. This yields

$$\begin{aligned}
\omega^{(1)} &= \tau^2 \\
\sqrt{Q^{(1)}} &= 1 + \frac{\omega^{(1)} - \left(\omega^{(1)}\right)^{-1}}{\sigma - \sigma^{-1}} \\
e_j^{(1,1)} &= I + \frac{\left(b_j^{+(1,1)} - b_j^{-(1,1)}\right)}{\sigma - \sigma^{-1}}.
\end{aligned} \tag{4.2}$$

The above relations together with the defining relations of the algebra (see section 3 and [9]) are sufficient to show that

$$\begin{aligned}
X_j(u) = p_j^{(1,1)} &+ \zeta^{-1}\eta^{-1}(z - z^{-1})\left(\tau^{-1} z\, b_j^{+(1,1)} - \tau z^{-1}\, b_j^{-(1,1)}\right) \\
&+ \eta^{-1}(\tau z^{-1} - \tau^{-1} z)\left(p_j^{(1,2)} + p_j^{(2,1)}\right) \\
&- \varepsilon_1 \zeta^{-1}\eta^{-1}(z - z^{-1})(\tau z^{-1} - \tau^{-1} z)\left(b_j^{(1,2)} + b_j^{(2,1)}\right) \\
&+ \varepsilon_2 \eta^{-1}(z - z^{-1})\left(e_j^{(1,2)} + e_j^{(2,1)}\right) \\
&+ \left(1 - \zeta^{-1}\eta^{-1}(z - z^{-1})(\tau z^{-1} - \tau^{-1} z)\right) p_j^{(2,2)}
\end{aligned} \tag{4.3}$$

satisfies the Yang–Baxter algebra (1.5). Here, the notation is the same as in section 2, i.e., $z = \exp(iu)$, $\zeta = (\sigma - \sigma^{-1})$, and $\eta = (\tau - \tau^{-1})$. Furthermore, $\varepsilon_1^2 = \varepsilon_2^2 = 1$ are two arbitrary signs. The appearance of this freedom is actually trivial since all relations of the dilute BWM algebra are invariant under the transformations $b_j^{\pm(a,b)} \to (-1)^{a-b} b_j^{\pm(a,b)}$ and $e_j^{(a,b)} \to (-1)^{a-b} e_j^{(a,b)}$.

The expression (4.3) is manifestly crossing-symmetric with crossing parameter λ defined by $\tau = \exp(i\lambda)$. Note that in order to have the crossing transformations of the braid and monoid operators as suggested by the diagrammatic interpretation (see [9]) one should use $\varepsilon_2 = 1$ in equation (4.3). This stems from the fact that the mixed monoid operators $e_j^{(a,b)}$ are crossing-related to the mixed projectors $p_j^{(a,b)}$ $(a \neq b)$ which have a fixed sign due to the requirement that the sum of the projectors gives the identity. The inversion relation (2.4) is satisfied by (4.3) with

$$\varrho(u) = \zeta^{-1}\eta^{-1}(\sigma z^{-1} - \sigma^{-1} z)(\tau z^{-1} - \tau^{-1} z) \tag{4.4}$$

which formal'y coincides with equation (2.5).

Comparing the above expression (4.3) with equation (2.3), one observes that not only the inversion relation but also the part which only involves colour '1' has exactly the same form as for the pure Birman–Wenzl–Murakami case. But in both cases one has to keep in mind that for a given representation, τ (and hence η) has a different meaning in the two expressions (2.3) and (4.3), because the twist is given by $\omega^{(1)} = \tau^2$ here whereas $\omega = \sigma\tau^2$ in the discussion of section 2. Obviously, the colour-'1' part of equation (4.3) alone does not satisfy a Yang–Baxter equation.

Alternatively, equation (4.3) (with $\varepsilon_1 = 1$) can be expressed in a more 'symmetric' form which reads

$$\begin{aligned} X_j(u) = \eta^{-1}(\tau z^{-1} - \tau^{-1} z) I \\ -\zeta^{-1}\eta^{-1}(z^{1/2} - z^{-1/2})(\tau z^{-1} - \tau^{-1} z)\left(z^{1/2} B_j + z^{-1/2} B_j^{-1}\right) \\ +\eta^{-1}(z^{1/2} - z^{-1/2})(\tau z^{-1/2} + \tau^{-1} z^{1/2})\left(e_j^{(1,1)} + e_j^{(2,2)}\right) \\ +\varepsilon_2 \eta^{-1}(z - z^{-1})\left(e_j^{(1,2)} + e_j^{(2,1)}\right) \end{aligned} \tag{4.5}$$

where we used the same notation as in equation (4.3) and

$$\begin{aligned} I = p_j^{(1,1)} + p_j^{(1,2)} + p_j^{(2,1)} + p_j^{(2,2)} \\ B_j^{\pm 1} = b_j^{\pm(1,1)} + b_j^{\pm(1,2)} + b_j^{\pm(2,1)} + b_j^{\pm(2,2)}. \end{aligned} \tag{4.6}$$

We include this second form since it treats both colours on an equal footing and might be more suitable for possible generalizations.

As in section 2, exchanging $\sigma \leftrightarrow -\sigma^{-1}$ leaves all algebraic expressions invariant. But contrary to the former case, this does not lead to a different solution as the value of τ (defined by $\omega^{(1)} = \tau^2$) is not affected by this transformation and hence the Yang–Baxter operator is also unchanged.

The remainder of this section deals with the $B_n^{(1)}$ and $D_{n+1}^{(2)}$ vertex models. This follows a dual purpose: on one hand we want to show that the $D_{n+1}^{(2)}$ models provide examples for the algebraic structure defined above, on the other hand we shall see that the representations corresponding to the $D_{n+1}^{(2)}$ vertex models can easily be obtained from those related to the $B_n^{(1)}$ vertex models. The reason why this is important is simply that the same procedure should work for face models also, at least in the trigonometric case.

Let us commence with the braid–monoid algebra representation related to the $B_n^{(1)}$ vertex models. We define

$$\begin{aligned} b_j^{\pm 1} = I \otimes I \otimes \ldots \otimes I \otimes b^{\pm 1} \otimes I \otimes \ldots \otimes I \otimes I \\ e_j = I \otimes I \otimes \ldots \otimes I \otimes e \otimes I \otimes \ldots \otimes I \otimes I \end{aligned} \tag{4.7}$$

where $b^{\pm 1}$ and e act at the two positions j and $j+1$ and I denotes the identity in one factor. Using the notation of [19], the explicit form of the $d^2 \times d^2$ ($d = 2n+1$) matrices $b^{\pm 1}$ and e reads

$$\begin{aligned} b = \sum_{\alpha} k^{-1}\left(1 + (k-1)\,\delta_{\alpha,\alpha'}\right) E_{\alpha,\alpha} \otimes E_{\alpha,\alpha} + \sum_{\alpha \neq \beta}\left(1 + (k-1)\,\delta_{\alpha,\beta'}\right) E_{\alpha,\beta} \otimes E_{\beta,\alpha} \\ -(k - k^{-1}) \sum_{\alpha<\beta} E_{\alpha,\alpha} \otimes E_{\beta,\beta} + (k - k^{-1}) \sum_{\alpha>\beta} k^{\bar\alpha - \bar\beta} E_{\alpha',\beta} \otimes E_{\alpha,\beta'} \end{aligned} \tag{4.8}$$

$$\begin{aligned} b^{-1} = \sum_{\alpha} k\left(1 + (k^{-1}-1)\,\delta_{\alpha,\alpha'}\right) E_{\alpha,\alpha} \otimes E_{\alpha,\alpha} + \sum_{\alpha \neq \beta}\left(1 + (k^{-1}-1)\,\delta_{\alpha,\beta'}\right) E_{\alpha,\beta} \otimes E_{\beta,\alpha} \\ +(k - k^{-1}) \sum_{\alpha>\beta} E_{\alpha,\alpha} \otimes E_{\beta,\beta} - (k - k^{-1}) \sum_{\alpha<\beta} k^{\bar\alpha - \bar\beta} E_{\alpha',\beta} \otimes E_{\alpha,\beta'} \end{aligned} \tag{4.9}$$

$$e = k^{2n-1} \sum_{\alpha,\beta} k^{\bar\alpha - \bar\beta} E_{\alpha',\beta} \otimes E_{\alpha,\beta'}. \tag{4.10}$$

Here, $1 \leqslant \alpha, \beta \leqslant d$, $\alpha' = d + 1 - \alpha$ ($d = 2n+1$), and $\bar\alpha$ is defined by

$$\bar\alpha = \begin{cases} \alpha + \frac{1}{2} & 1 \leqslant \alpha \leqslant n \\ \alpha & \alpha = n+1 \\ \alpha - \frac{1}{2} & n+2 \leqslant \alpha \leqslant 2n+1. \end{cases} \tag{4.11}$$

Besides, $E_{\alpha,\beta}$ are $d \times d$ matrices with elements $(E_{\alpha,\beta})_{i,j} = \delta_{i,\alpha}\delta_{j,\beta}$.

The matrices defined above satisfy the equations

$$(b - k^{-1}I)\,(b + kI)\,(b - k^{2n}I) = 0 \tag{4.12}$$

and

$$e = I + \frac{b - b^{-1}}{k - k^{-1}}. \tag{4.13}$$

Hence, $b_j^{\pm 1}$ and e_j (4.7) form a representation of the BWM algebra with

$$\omega = k^{2n} \qquad \sqrt{Q} = 1 + \frac{k^{2n} - k^{-2n}}{k - k^{-1}}. \tag{4.14}$$

The corresponding Yang–Baxter operator (2.3) with $\sigma = k$ and $\tau = k^{n-1/2}$ yields exactly the R matrix of [19] (with $x = z^2$).

In order to obtain a representation of the dilute BWM algebra, we add one extra state to the local spaces, which is going to correspond to the second colour. The corresponding matrices which act on the tensor product of two spaces now have the dimension $(d+1)^2 \times (d+1)^2$. The (two-site) projectors $p^{(a,b)}$ are given by $p^{(a,b)} = P^{(a)} \otimes P^{(b)}$ with

$$\begin{aligned} P^{(1)} &= \sum_\alpha E_{\alpha,\alpha} \\ P^{(2)} &= E_{d+1,d+1} \end{aligned} \tag{4.15}$$

where the summation variables here and in what follows are always restricted to the values $1 \leqslant \alpha, \beta \leqslant d$ which correspond to the states of colour '1' and the now $(d+1) \times (d+1)$ matrices $E_{\alpha,\beta}$ are defined as above. The representation matrices for the BWM part (which is the part that involves colour '1' only) $b^{\pm(1,1)}$ and $e^{(1,1)}$ are given by the same expressions as the matrices $b^{\pm 1}$ (equations (4.8) and (4.9)) and e (4.10), respectively, but of course they are now $(d+1)^2 \times (d+1)^2$ matrices as well. The mixed braids $b^{(1,2)}$ and $b^{(2,1)}$ are

$$\begin{aligned} b^{(1,2)} &= \sum_\alpha E_{d+1,\alpha} \otimes E_{\alpha,d+1} \\ b^{(2,1)} &= \sum_\alpha E_{\alpha,d+1} \otimes E_{d+1,\alpha} \end{aligned} \tag{4.16}$$

and the mixed Temperley–Lieb operators have the form

$$\begin{aligned} e^{(1,2)} &= -k^{n+1} \sum_\alpha k^{-\bar{\alpha}} E_{d+1,\alpha} \otimes E_{d+1,\alpha'} \\ e^{(2,1)} &= -k^{-(n+1)} \sum_\alpha k^{\bar{\alpha}} E_{\alpha',d+1} \otimes E_{\alpha,d+1} \end{aligned} \tag{4.17}$$

with α' and $\bar{\alpha}$ defined as before.

The above equations define a representation of the dilute BWM algebra with

$$\begin{aligned} \omega^{(1)} &= k^{2n} \\ \sqrt{Q^{(1)}} &= 1 + \frac{k^{2n} - k^{-2n}}{k - k^{-1}} \\ \omega^{(2)} &= \sqrt{Q^{(2)}} = 1 \end{aligned} \tag{4.18}$$

Correspondingly, we obtain a representation of the Yang–Baxter algebra via equations (4.3) or (4.5) with $\sigma = k$ and $\tau = k^n$ and thereby have a solvable vertex model with $2n+2$ states.

As it turns out, these models are just the $D^{(2)}_{n+1}$ vertex models which play a somewhat singular role in [19] as they are the only series of R matrices which do not commute at one place. This means that in general $[\check{R}(u), \check{R}(v)] \neq 0$ which already implies that the Yang–Baxter operator cannot be written as a polynomial in a braid operator alone. The expression obtained here (choosing $\varepsilon_1 = \varepsilon_2 = 1$ in equations (4.3) or (4.5)) is related to $\check{R}(x) = PR(x)$ of [19] (with $x = z$, $\xi = k^n = \tau$) by an orthogonal transformation with the matrix $S \otimes S$ where $S = S_1 \cdot S_2$ and

$$\begin{aligned} S_1 &= \sum_{\alpha=1}^{n+1} E_{\alpha,\alpha} + \sum_{\alpha=n+2}^{d} E_{\alpha,\alpha+1} + E_{d+1,n+2} \\ S_2 &= \sum_{\alpha=1}^{n} E_{\alpha,\alpha} + \sum_{\alpha=n+3}^{d+1} E_{\alpha,\alpha} + \frac{1}{\sqrt{2}} \left(E_{n+1,n+1} + E_{n+1,n+2} + E_{n+2,n+1} - E_{n+2,n+2} \right). \end{aligned} \tag{4.19}$$

In other words, the additional colour-'2' state corresponds in Jimbo's basis [19] to the asymmetric combination of states $n+1$ and $n+2$. In particular, the projectors onto the two colours are not diagonal in that basis.

5. Summary and outlook

A 'dilute' Birman–Wenzl–Murakami algebra has been defined as a generalization of the well known BWM algebra [6, 7]. This was done following the general ideas of [9] on multi-colour braid–monoid algebras. Similar to the Birman–Wenzl–Murakami case, the dilute algebra can be Baxterized to a solution of the Yang–Baxter algebra. This means that every appropriate matrix representation of the dilute algebra defines a solvable lattice model.

As an example, the representation of the BWM algebra which corresponds to the $B_n^{(1)}$ vertex models was considered explicitly and was enlarged to a representation of the dilute algebra. It turned out that the solvable vertex models obtained from this representation are the $D_{n+1}^{(2)}$ vertex models, the R matrix differing from that of [19] only by a simple similarity transformation.

There are a number of questions raised by the results of this paper.

The first, of course, is about the nature of the 'second' series of solutions related to the $C_n^{(1)}$ representations of the BWM algebra (see the last paragraph of section 2). It appears that these correspond to another series of $A_{2n-1}^{(2)}$ models, the two series of $A_{2n-1}^{(2)}$ models being related to different realizations of the twisted affine Lie algebra $A_{2n-1}^{(2)}$ (compare the comments in [22] at the end of section 1). These models are discussed in more detail in [25].

Another question concerns face models related to the $D_{n+1}^{(2)}$ vertex models. The result of section 4 means that one can construct such models (at least with trigonometric weights) on the basis of the known $B_n^{(1)}$ models [26], in a similar way as the dilute A–D–E models [10–12, 9] are related to the 'usual' (i.e. non-dilute) A–D–E models (see [27] and references therein). To do this, one has to find a dilute extension of the corresponding representations of the braid–monoid algebra. As in the case of the dilute A–D–E models [11], one might expect that these models can be extended away from criticality to yield interesting elliptic solutions to the Yang–Baxter equation. This will be the subject of a future publication [28].

Equations (4.3) or (4.5) give a solution of the Yang–Baxter equation for any representation of the dilute BWM algebra. But if one has a representation of the BWM algebra itself it appears to be quite straightforward to generalize it to the dilute case. This can be seen in the example of the $D_{n+1}^{(2)}$ models in section 4 which can be constructed starting from the BWM representation provided by the $B_n^{(1)}$ models. On the other hand, we had three such series in section 2, the other two being related to the $C_n^{(1)}$ and $D_n^{(1)}$ models. Apparently, these will also give rise to corresponding series of dilute models which at first glance do not seem to fit into the list of known solvable models. But even if they in fact are related to known models (for instance, by a gauge transformation) these expressions might still be of use. It is plausible that—as it happens in other cases (for example, for the $A_2^{(2)}$ models [22, 10, 11], see also [12, 29])—there exist several series of non-equivalent solvable face models which are related to the vertex model R matrix in different gauges. These questions are currently being investigated and the results will be presented in forthcoming publications [30, 25, 28].

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